A five-fold equivalence in special relativity one-dimensional dynamics

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Abstract

Conservation of energy, as shown by Coleman (2005 *Eur. J. Phys.* **26** 647-50), leads by virtue of little appreciated identities in a one-dimensional context to equality of particle-frame and 'equivalent' observer-frame forces (Newton's third law of motion relativistically 'upgraded'), to conservation of 'spatial momentum', to conservation of 'temporal momentum' (so-called 'relativistic mass') and to kinetic energy formula $K = mc^2(\gamma - 1)$. As easily shown, postulating any one of these five principles validates all the others.

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We recall [1] identities from velocity composition and time dilation $\alpha \equiv a\gamma^3 \equiv d(\gamma v)/dt \equiv c^2 d(\gamma)/dx$, where α and a are the particle-frame (χ, τ) and observer-frame (x,t) accelerations respectively, as well as definitions 'spatial momentum' $p_x \triangleq mv\gamma \equiv mdx/d\tau$, 'temporal momentum'[2] $p_t \triangleq m\gamma \equiv mdt/d\tau$ and K as *the kinetic energy imparted to the particle*. To these were added the definitions

EQUIVALENT OBSERVER-FRAME APPLIED FORCE

$$F_x \triangleq \Delta K_x/\Delta x, \qquad (A)$$

and PARTICLE-EXPERIENCED FORCE

$$\Phi_x = ma = m\alpha\gamma^3 \equiv m\frac{d(v\gamma)}{dt} \triangleq \frac{dp_x}{dt} \equiv mc^2\frac{d\gamma}{dx} \triangleq c^2\frac{dp_t}{dx}. \qquad (B)$$

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[2] 'Temporal momentum' $m\gamma = mdt/d\tau$ replaces the historic misnomer 'relativistic mass' [1, 3].

Considering that *"many elementary textbooks fail to present a valid derivation of the [mass-energy] equation"* [2], it is noteworthy that on the basis of definitions (A) and (B), the main edifice of single-dimension special relativity dynamics can be established by postulation of *any one* of five principles:

(I) *Conservation of energy*, i.e. the incremental *imparted* kinetic energy dK, equals the particles *acquired* kinetic energy incremental term in definition (B):
$dK = c^2 dp_t \triangleq c^2 md\gamma.$

(II) Total acquired particle kinetic energy relates as
$K = \int_{v=0}^{v=v} c^2 md\gamma = mc^2(\gamma - 1).$

(III) 'Equivalent' observer-applied force equals particle-experienced force, i.e. Newtons third law of motion can be applied in a generalized relativistic context:
$F_x = \Phi_x$.
The final two principles involve values before and after a two-particle collision:

(IV) *Conservation of spatial momentum*: $\int d(p_{1x} + p_{2x}) = 0$, i.e. $\int dt \frac{dp_{1x}}{dt} = \frac{-dp_{2x}}{dt}$,

(V) *Conservation of temporal momentum*: $\int d(p_{1t} + p_{2t}) = 0$, i.e. $\int dx \frac{dp_{1t}}{dx} = \frac{-dp_{2t}}{dx}$.

(I) directly implies (II) and vice versa. (I)—with force definitions (A) and (B)—also implies (III), just as (III)—with (A) and (B)—implies (I). Hence (III) is equivalent to both (I) and (II). Each of (IV) and (V) follows from (III) and vice versa. Therefore, in conjunction with force *definitions* (A) and (B), any one of the statements (I)-(V) is a necessary and sufficient condition for each one of the others. All five principles—conceptualizations of special relativity dynamics in one spatial dimension—are equivalent.

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References

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