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Special relativity dynamics without *a priori* momentum conservation

Abstract [1] Single-spatial dimension intrinsic identities involving a particle's own-acceleration and observer-perceived acceleration, permit *'spatial momentum'*—a parameter defined as $m.dx/d\tau$—and its spatio-temporal complement $m.dt/d\tau$ re-labelled *'temporal momentum'* in place of the misnomer *'relativistic mass'*—to reflect forces traditional link between classical momentum and kinetic energy. Energy conservation confirms the relativistic force parallel and leads directly to the mass-energy formula.

Expositions of relativistic dynamics are usually vectorial and generally invoke conservation of momentum. Exceptions to this are seldom referenced papers Ehlers et al [2] and Simon and Husson [4]—which both relate to a rediscovered 1920 derivation by the French physicist Paul Langevin [1]—and recently Sonego and Pin [11]. On a notably more elementary level, requiring merely standard identities obtained from the velocity composition equation, a straightforward single-spatial dimension Gedankenexperiment (thought experiment) directly establishes the fundamental relativistic expression for mass and energy. Conservation of momenta is also an immediate consequence.

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Familiar spatio-temporal relationships

For a particle accelerating at rate α in its momentarily co-moving reference frame, with own- time[2] τ and coordinates $(x,t\ )$ in the reference frame of an arbitrary non-accelerating observer, velocity is $v = dx/dt$. With time dilation factor $\gamma \triangleq \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}}$, velocity composition[8] $v+\Delta v = \frac{v+\alpha\Delta\tau}{1+v.\alpha\Delta\tau/c^2}$ gives $\frac{dv}{d\tau} = \frac{\alpha}{\gamma^2}$. Hence $a \triangleq \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{d\tau}\frac{d\tau}{dt} = \frac{\alpha}{\gamma^3}$. A particle's own-acceleration thus equals its observer-perceived acceleration multiplied by γ^3. As is also well known (simple proof below[3]), α is symmetrically related to $v\gamma$ and $c^2\gamma$:

$$\alpha = a\gamma^3 \equiv \frac{d(v\gamma)}{dt} \equiv \frac{c^2 d\gamma}{dx} \tag{1}$$

A definition of relativistic force

Taking the particles mass m as a constant, we use (1) to define three parameters: *relativistic force* Φ_x, *spatial momentum* p_x and *temporal momentum* p_t:

$$\Phi_x \triangleq m\alpha = ma\gamma^3 = m\frac{d(v\gamma)}{dt} \triangleq \frac{dp_x}{dt} = mc^2\frac{d\gamma}{dx} \triangleq c^2\frac{dp_t}{dx}. \tag{2}$$

Temporal momentum (an expression found in *quantum mechanics* [9]) i.e. $p_t = m\gamma = m.dt/d\tau$, being proportional to a 't-direction' *speed of time* $dt/d\tau$, is arguably therefore a 'natural' spatio-temporal complement of spatial momentum $p_x = mv\gamma = mdx/d\tau$. The former also happens to constitute what is traditionally known as 'relativistic mass' (nowadays largely discarded as a misnomer [6]).

Clearly our defined relativistic force corresponds—for fixed m—to the classical physical force (as expressed in Newton's second law of motion) experienced by the particle in its own momentary space-time frame, where $v = 0$ and $\gamma = 1$.

Linking 'experienced force' to 'equivalent applied force'

We now deploy the *Gedankenexperiment* method by imagining an arbitrary agent transferring an increment of kinetic energy ΔK_x to the particle as it moves over an incremental distance x—*viewed from the observer spacetime frame*. The transfer is considered perfectly 'elastic' i.e. all the energy imparted contributes to the particle's ongoing acceleration at a rate equal to α *as judged in the particle's*

[2] Avoiding the adjective *'proper'*—a mistranslation of the French 'propre'—whose usage in relativity is *dysfunctional* in that it inappropriately suggests *'correctness'* and fails to signify the crucial meaning of *attachment*.

[3] Putting $q = 1 - \frac{v^2}{c^2}$ gives $\frac{-c^2}{vdv} = \frac{2}{dq}$ and therefore (since $\frac{d(1/q^{1/2})}{d(q^{1/2})} = -(q^{1/2})^{-2} = -q^{-1}$):
$\frac{c^2 d\gamma}{dx} = \frac{c^2 d(1/q^{1/2})}{dx} = \frac{-c^2 d(q^{1/2})}{qdx}\frac{adt}{dv} = \frac{-ac^2 d(q^{1/2})}{qvdv} = \frac{2adq^{1/2}}{qdq} = \frac{a}{q^{3/2}} = a\gamma^3$.
Also $\frac{d(v\gamma)}{dt}\frac{dx}{dx} = \frac{vd(v\gamma)}{dx} = \frac{v^2 d\gamma}{dx} + \frac{v\gamma dv}{dx}\frac{dt}{dt} = \frac{v^2 a\gamma^3}{c^2} + a\gamma = a\gamma\left(\frac{v^2/c^2}{1-v^2/c^2} + 1\right) = a\gamma^3$.

own spacetime frame. Accordingly we may further define *'an equivalent observer spacetime force'*:

$$F_x \triangleq \Delta K_x/\Delta x. \tag{3}$$

Remark: This *'perceivable'* equivalent force F_x need not actually exist as a *force physically embodied in the observer space-time frame*. A rocket, for example, would be accelerated by fuelled propulsionan 'arbitrary agent' whose force in reality is always exerted *in the rocket space-time frame*. Physically relevant is *an equivalent cause*—$F_x\Delta x$—leading to *a resulting effect*—transfer of kinetic energy ΔK_x.

Calling upon the principle of *conservation of energy*, we consider the agent and particle as *a closed system*. Since the relativistic process is governed by relationships (2), the energy gain $c^2\Delta p_t$ associated—*from the viewpoint of the observer*—with the particle, must equal ΔK_x. Hence from (3):

$$F_x\Delta x = \Delta K_x = c^2\Delta p_t = \Phi_x\Delta x. \tag{4}$$

Consequently we arrive at what is conceivably a relativistic extension of Newton's third law of motion:

$$F_x = \Phi_x. \tag{5}$$

Equivalent applied force F_x perceivable as $mc^2d\gamma/dx = md(v\gamma)/dt$ in the observer space-time frame, equals experienced force Φ_x perceivable as $m\alpha$ in the particle's co-moving space-time frame.

The mass-energy relationship

The kinetic energy of the particle imparted over an arbitrary observer distance and inherent to the observer space-time frame, can be now readily obtained from (4) and (2):

$$K_x = \int_{v=0}^{v=v} \Phi_x dx = mc^2 \int_{v=0}^{v=v} d\gamma = mc^2\gamma - mc^2. \tag{6}$$

The *'perforce'* diversification of classical momentum into spatial and temporal momenta thus allows us to declare the familiar observer space-time mass-energy relationship:

$$E \triangleq c^2 p_t = mc^2\gamma. \tag{7}$$

which in the particle space-time frame ($v = 0$, $\gamma = 1$) is:

$$E_0 \triangleq c^2 p_{t0} = mc^2. \tag{8}$$

Momenta conservation

The expected observer-independent invariant for the momenta emerges directly from the actual definitions of p_x and p_t:

$$c^2pt^2 - p_x^2 = \frac{m^2c^2}{1 - v^2/c^2} - \frac{m^2c^2.v^2/c^2}{1 - v^2/c^2} = m^2c^2. \tag{9}$$

Finally, viewed from any observer space-time frame, two particles colliding elastically[4] will—by virtue of equality (5)—exert equal and opposite 'equivalent forces' upon one another. We can therefore write from (2):

$$\frac{dp_{1x}}{dt} = \frac{-dp_{2x}}{dt} \text{ i.e. } \int d(p_{1x} + p_{2x}) = 0 \int dt. \qquad (10)$$

Likewise:

$$\frac{c^2 dp_{1t}}{dx} = \frac{-dp_{2t}}{dx} \text{ i.e. } c^2 \int d(p_{1t} + p_{2t}) = 0 \int dx. \qquad (11)$$

Conclusion

On the basis of energy and mass conservation, relativistic velocity composition alone leads—with minimal effort—both to the kernel special relativity dynamics equations as well as to *deduced* conservation of spatial and temporal momenta.

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[4] After non-elastic collisions, in the merged particles' spacetime frame $p'_{1t} + p2_{1t} = m_1 + m_2 < p_{1t} + p_{2t}$. The overall temporal momentum balance is maintained however by the increased (on average symmetric) internal movement of the constituent molecules resulting from kinetic energy imparted as heat.